

KOLMOGOROV COMPACTNESS CRITERION IN VARIABLE EXPONENT LEBESGUE SPACES

HUMBERTO RAFEIRO

ABSTRACT. The well-known Kolmogorov compactness criterion is extended to the case of variable exponent Lebesgue spaces $L^{p(\cdot)}(\overline{\Omega})$, where Ω is a bounded open set in \mathbb{R}^n and $p(\cdot)$ satisfies some “standard” conditions. Our final result should be called Kolmogorov-Tulajkov-Sudakov compactness criterion, since it includes the case $p_- = 1$ and requires only the “uniform” condition.

1. INTRODUCTION

We extend Kolmogorov-Tulajkov-Sudakov criterion to the case of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$. The theory of the variable exponent Lebesgue spaces was intensively developed during the last two decades, inspired both by difficult open problems in this theory, and possible applications shown in [9], we refer e.g. to the surveying papers [2, 4, 11] on the topic and references therein.

The classical theorem of Kolmogorov [5](see also, e.g., [8]) about compactness of subsets in L^p , can be stated in the following terms

Theorem (Kolmogorov). *Suppose \mathfrak{F} is a set of functions in $L^p([0, 1])$ ($1 < p < \infty$). In order that this set be relatively compact, it is necessary and sufficient that both of the following conditions be satisfied:*

- (k1) *the set \mathfrak{F} is bounded in L^p ;*
- (k2) $\lim_{h \rightarrow 0} \|f_h - f\|_p = 0$ *uniformly with respect to $f \in \mathfrak{F}$,*

where f_h denotes the well-known Steklov function

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt.$$

After that, Tamarkin [14] extended the result to the case where the underlying space can be unbounded, with an additional condition related to behavior at infinity. Tulajkov [15] showed that Tamarkin’s result was true even when $p = 1$. Finally, Sudakov [12] showed that condition (k1) follows from condition (k2).

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Kolmogorov's compactness criterion has also been extended to other function spaces, for example, it was extended by Takahashi [13] for *Orlicz spaces* satisfying the Δ_2 -condition, by Goes and Welland [3] for *continuously regular Köthe spaces* and by Musielak [7] to *Musielak-Orlicz spaces*, just to mention a few.

The main result of this paper is given in Theorems 11 and 15. Although it seems that we can obtain our Theorems 11 and 15 from [3], we give its straightforward proof, which is easier than to derive it from the multi-step proof in [3]. The direct proof we suggest is completely within the frameworks of variable exponent Lebesgue spaces.

A version of the proof of Theorem 11 that appears in [7, p. 63] in the context of Musielak-Orlicz spaces, uses an extra condition, the so-called ∞ -condition (see [7, p. 61]), but this condition is not satisfied even in the case of constant $p = 1$, while our proof admits the case $\inf p(x) = 1$.

2. PRELIMINARIES

We refer to papers [6, 10] and surveys [2, 4, 11] for details on variable Lebesgue spaces over domains in \mathbb{R}^n , but give some necessary definitions. Let $\Omega \subseteq \mathbb{R}^n$ be an open set in \mathbb{R}^n . By $L^{p(\cdot)}(\Omega)$ we denote the space of functions $f(x)$ on Ω such that

$$I_p(f) = \int_{\Omega} |f(x)|^{p(x)} dx$$

where $p(x)$ is a measurable function on Ω with values in $[1, \infty)$ and define $p_- = \text{ess inf}_{x \in \Omega} p(x)$ and $p_+ = \text{ess sup}_{x \in \Omega} p(x)$. This is a Banach space with respect to the norm

$$(1) \quad \|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : I_p \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

We list the following properties of the space $L^{p(\cdot)}(\Omega)$ that will be needed:

(a) *Hölder's inequality:*

$$(2) \quad \int_{\Omega} |f(x)g(x)| dx \leq k \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}$$

where $1 \leq p(x) \leq \infty$, $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$, $k = \sup_{x \in \Omega} \frac{1}{p(x)} + \sup_{x \in \Omega} \frac{1}{q(x)}$.

(b) *estimates for the norm of the characteristic function of a set:*

$$(3) \quad |E|^{1/p_+} \leq \|\chi_E\|_{p(\cdot)} \leq |E|^{1/p_-}, \quad \text{if } |E| \leq 1,$$

the signs of the inequalities being opposite if $|E| \geq 1$.

(c) *denseness of step functions:* in the case $p_+ < \infty$, functions of the form $\sum_{k=1}^m c_k \chi_{\Omega_k}$, $\Omega_k \subset \Omega$, $|\Omega_k| < \infty$, with constant c_k , form a dense set in $L^{p(\cdot)}(\Omega)$.

(d) *denseness of continuous functions:* in the case $p_+ < \infty$, the set of continuous functions with a finite support is dense in $L^{p(\cdot)}(\Omega)$.

- (e) $L^{p(\cdot)}(\Omega)$ is ideal: i.e. it is complete and the inequality $|f| \leq |g|$ implies $\|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)}$.

Standard conditions in the framework of variable Lebesgue space are

$$(4) \quad |p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2},$$

and

$$(5) \quad |p(x) - p(\infty)| \leq \frac{C}{\ln(2+|x|)}.$$

Definition 1. By $w\text{-}Lip(\Omega)$ we denote the class of exponents $p \in L^\infty(\Omega)$ satisfying the (local) logarithmic condition (4).

Definition 2. By $\mathcal{P}_\infty(\Omega)$ we denote the class of exponents $p \in L^\infty(\Omega)$ which satisfy the condition that there exists $p(\infty) = \lim_{\Omega \ni x \rightarrow \infty} p(x)$ and the assumption (5).

2.1. Approximate identities. Let ϕ be an integrable function which have $\int_{\mathbf{R}^n} \phi(x) dx = 1$. For each $t > 0$, we put $\phi_t := t^{-n} \phi(t^{-1}x)$. Following [1], we say that $\{\phi_t\}$ is a potential-type approximate identity, if the radial majorant of ϕ , defined by

$$\tilde{\phi}(x) = \sup_{|y| \geq |x|} |\phi(y)|$$

is integrable. In [1] the following proposition was proved.

Proposition 3. *Given an open set Ω , let $p \in \mathcal{P}_\infty(\Omega) \cap w\text{-}Lip(\Omega)$. If $\{\phi_t\}$ is a potential-type approximate identity, then for all $t > 0$, we have:*

- (i) $\|\phi_t * f\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)},$
- (ii) $\lim_{t \rightarrow 0} \|\phi_t * f - f\|_{p(\cdot)} = 0.$

The following propositions are well-known, and can be found in standard books dealing with the subject of compactness, they are stated for the sake of completeness.

Definition 4. Let (X, d) be a metric space, X_0 a subspace of X , and $\varepsilon > 0$. A subset N in X is called a *Hausdorff ε -net* (or just ε -net) for X_0 if for each element $x \in X_0$ there is a $x_\varepsilon \in N$ such that $d(x, x_\varepsilon) < \varepsilon$.

Definition 5. A subset X_0 of a metric space X is called *totally bounded* if for every $\varepsilon > 0$ in X there is a finite (i.e., consisting of finitely many elements) ε -net for X_0 .

Proposition 6. *A subset of a complete metric space is relatively compact (i.e., its closure is compact) if and only if it is totally bounded.*

Definition 7. Let X and Y be metric spaces. A family \mathcal{F} of functions $f : X \rightarrow Y$ is *equicontinuous on X* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ for any function $f \in \mathcal{F}$ and all $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$.

Definition 8. The functions of a set \mathcal{F} are said to be *uniformly bounded* if $\sup_{f \in \mathcal{F}} \sup_{x \in X} |f(x)| < \infty$.

Proposition 9 (Ascoli-Arzelà). *Let Ω be a bounded open set in \mathbb{R}^n . A subset K of $C(\overline{\Omega})$ is relatively compact in $C(\overline{\Omega})$ if K is equicontinuous and is also uniformly bounded.*

3. COMPACTNESS CRITERION

3.1. Steklov function. We will use both f_h or $\Phi_h * f$ to denote the Steklov function

$$(6) \quad f_h(x) = \Phi_h * f(x) = \frac{1}{v_n h^n} \int_{B(x,h)} f(t) \, dt$$

where $h > 0$, $\Phi(x) = \frac{1}{v_n} \chi_{B(0,1)}(x)$, $\Phi_h(x) = \frac{1}{h^n} \Phi\left(\frac{x}{h}\right)$, $v_n = |B(0,1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$, and it is assumed that the function $f(x)$ is continued beyond Ω as identical zero, whenever necessary.

Lemma 10. *Given an open set Ω , let $p \in \mathcal{P}_\infty(\Omega) \cap w\text{-Lip}(\Omega)$. Then*

- (i) $\|f_h\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}$;
- (ii) $\lim_{h \rightarrow 0} \|f_h - f\|_{p(\cdot)} = 0$.

where $C > 0$ does not depend on $h > 0$ and $f \in L^{p(\cdot)}(\Omega)$.

Proof. It suffices to note that $\{\Phi_h\}$ is a potential-type approximate identity and apply Proposition 3. \square

3.2. Kolmogorov compactness criterion. The following result is an extension to the variable Lebesgue spaces of the well known Kolmogorov compactness criterion (or, to be more precise, Kolmogorov-Tulajkov compactness criterion since we admit $p_- = 1$).

Theorem 11. *Let \mathfrak{F} be a subset of $L^{p(\cdot)}(\overline{\Omega})$, where Ω is a bounded open set in \mathbb{R}^n and let $p \in \mathcal{P}_\infty(\overline{\Omega}) \cap w\text{-Lip}(\overline{\Omega})$. The set \mathfrak{F} is relatively compact if and only if the following conditions are satisfied:*

- (i) $\lim_{h \rightarrow 0} \|f_h - f\|_{p(\cdot)} = 0$ uniformly for $f \in \mathfrak{F}$,
- (ii) \mathfrak{F} is bounded in $L^{p(\cdot)}(\overline{\Omega})$.

Proof. Necessity.

We only have to prove the equicontinuity (i). By Proposition 6, for every $\varepsilon > 0$, there exists a finite ε -net for the set \mathfrak{F} . Since step functions are dense in $L^{p(\cdot)}(K)$, the finite ε -net can be chosen as the set $\{s_j\}_{j=1}^\ell$ of simple functions $s_j(x)$ with the property

$$(7) \quad \|f - s_j\|_{p(\cdot)} < \varepsilon.$$

To prove (i), we first note that, by Lemma 10(ii), given $\varepsilon > 0$, there exists an h_j indexed to each s_j such that

$$\|\Phi_h * s_j - s_j\|_{p(\cdot)} < \varepsilon$$

whenever $h < h_j$. Letting $h_0 = \min_{1 \leq j \leq \ell} h_j$, we have

$$\|\Phi_h * s_j - s_j\|_{p(\cdot)} < \varepsilon$$

for all $j = 1, \dots, \ell$ whenever $h < h_0$.

Then for $h < h_0$ and all $f \in \mathfrak{F}$ we have a suitable s_r such that

$$\begin{aligned} \|\Phi_h * f - f\|_{p(\cdot)} &\leq \|\Phi_h * f - \Phi_h * s_r\|_{p(\cdot)} + \|s_r - f\|_{p(\cdot)} + \|\Phi_h * s_r - s_r\|_{p(\cdot)} \\ &\leq (C + 1)\|f - s_r\|_{p(\cdot)} + \varepsilon = (C + 2)\varepsilon \end{aligned}$$

where C is from Lemma 10, which gives the necessity of (i).

Sufficiency.

Let $\mathfrak{F}_h = \{f_h : f \in \mathfrak{F}\}$, where f_h is the Steklov function (6). By Hölder's inequality (2) and property (3), we obtain

$$\begin{aligned} (8) \quad v_n h^n |f_h(x)| &\leq k \|f\|_{p(\cdot)} \|\chi_{B(x,h)}\|_{q(\cdot)} \\ &\leq kM(v_n h^n)^{1/q-} \end{aligned}$$

whenever $v_n h^n \leq 1$. This means that all the functions in \mathfrak{F}_h are uniformly bounded under suitable choice of h due to condition (ii).

Let us define another set, namely, $\mathfrak{F}_{hh} = \{f_h : f \in \mathfrak{F}_h\}$. Functions of \mathfrak{F}_{hh} are of the form

$$(9) \quad f_{hh}(x) = \frac{1}{v_n h^n} \int_{B(x,h)} f_h(t) dt.$$

By (8) and using the same considerations made above, we have that all the functions in \mathfrak{F}_{hh} are uniformly bounded.

We want also to show that they are equicontinuous. We have

$$\begin{aligned} (10) \quad v_n h^n |f_{hh}(x+u) - f_{hh}(x)| &\leq \int_{\Omega(x,u,h)} |f_h(t)| dt \\ &\leq kM(v_n h^n)^{1/q-1} |\Omega(x,u,h)| \end{aligned}$$

where $\Omega(x,u,h) = B(x+u,h) \Delta B(x,h)$ and Δ stands for the symmetric difference of sets.

Since $B(x+u,h) \subset B(x,|u|+h)$ and $B(x+u,h) \supset B(x,h-|u|)$ whenever $|u| \leq h$, we have that

$$(11) \quad \Theta(x,u,h) := B(x,|u|+h) \setminus B(x,h) \cup B(x,h) \setminus B(x,h-|u|) \supset \Omega(x,u,h).$$

The Lebesgue measure of Θ is given by

$$\begin{aligned} (12) \quad |\Theta(x,u,h)| &= v_n \{[(|u|+h)^n - h^n] + [h^n - (h-|u|)^n]\} \\ &\leq 2nv_n |u|(2h)^{n-1}, \quad \text{when } |u| \leq h. \end{aligned}$$

By (10)-(12) we have the inequality

$$|f_{hh}(x+u) - f_{hh}(x)| \leq C|u|$$

whenever $|u| \leq h$ and h is fixed and sufficiently small, which proves the equicontinuity of \mathfrak{F}_{hh} .

Thus, for an arbitrary fixed $h > 0$ such that $v_n h^n \leq 1$, the set \mathfrak{F}_{hh} is relatively compact in $C(\overline{\Omega})$ due to Ascoli-Arzelà proposition 9. By the fact that $C(\overline{\Omega})$ is dense in $L^{p(\cdot)}(\overline{\Omega})$ we can take an $\varepsilon/2$ approximation by continuous functions, and from this $\varepsilon/2$ approximation we can obtain a finite $\varepsilon/2$ -net of \mathfrak{F}_{hh} to $C(\overline{\Omega})$, entailing a finite ε -net of \mathfrak{F}_{hh} to $L^{p(\cdot)}(\overline{\Omega})$, which gives relative compactness by Proposition 6. Finally, relative compactness of \mathfrak{F} regarding $L^{p(\cdot)}(\overline{\Omega})$ follows from condition (i), relative compactness of \mathfrak{F}_{hh} and the same reasoning as above applied twice, first to \mathfrak{F}_h and finally to \mathfrak{F} . \square

The condition (ii) of Theorem 11 can be omitted, since it follows from condition (i), this was proved in [12] for the case of classical Lebesgue spaces, which is also valid for the case of variable exponent Lebesgue spaces, as shown in Lemma 14.

To this end, we need the following results, where Proposition 12 is a statement known in the Riesz-Schauder theory, see e.g. [16, pp. 283-286]

Proposition 12. *Let U be a linear operator with a compact power. If $\lambda_0 \neq 0$ is not an eigenvalue of U , then λ_0 is in the resolvent set of U , i.e., the operator $(U - \lambda_0 I)^{-1}$ is continuous.*

Lemma 13. *Let G be a subset of a normed space and, for some $K > 0$, we have that for all $x \in G$ $\|Ux - x\| < K$. Then for such x*

$$\|x\| \leq K\|(U - I)^{-1}\|$$

i.e., the set G proves to be bounded whenever $(U - I)^{-1}$ is a bounded operator.

We can now prove the following lemma.

Lemma 14. *Let \mathfrak{F} be a subset of $L^{p(\cdot)}(K)$, where K is a bounded closed set in \mathbb{R}^n and let $p \in \mathcal{P}_\infty(K) \cap w\text{-Lip}(K)$. Then condition (i) of Theorem 11 implies condition (ii).*

Proof. By Proposition 12 and Lemma 13, if $\lambda = 1$ is not an eigenvalue of U , the boundedness of G is a consequence of a uniform approximation of elements $x \in G$ by elements of Ux whenever U has some power of it being compact.

In our case, the operator U is given by the Steklov function, i.e., $Ux = \Phi_h * x$, it is compact in square in $L^{p(\cdot)}(K)$ (see the sufficiency part in the proof of Theorem 11) and the uniform approximation property is given by condition (i) of Theorem 11. Therefore we only need to show that $\lambda = 1$ cannot be an eigenvalue of the Steklov operator.

Let us suppose, for the sake of contradiction, that exists $L^{p(\cdot)}(K) \ni x(t) \neq 0$ such that

$$(13) \quad \Phi_h * x(t) = x(t) \quad \text{for } t \in K,$$

where we continue the function $x(t)$ by zero beyond K .

The function $\Phi_h * x(t)$ is continuous and vanishes beyond some bounded set. Being continuous attains its maximum and minimum and at the least one of them is different from 0.

Let us suppose that $M = \max_t \Phi_h * x(t) > 0$. We define P as the set on \mathbb{R}^n such that $\Phi_h x(t)$ attains its maximum. P is closed and bounded. Let us choose an arbitrary boundary point $t_0 \in \partial P \cap K$. Then the ball $B(t_0, h)$ contains a set of positive measure in K such that $x(t) = \Phi_h * x(t) < M$, this by (13) and the definition of P , but then $\Phi_h * x(t_0)$ cannot attain the maximum since it is an average of $x(t)$ inside $B(t_0, h)$. The obtained contradiction proves the statement. \square

By Theorem 11 and Lemma 14 we obtain

Theorem 15. *Let \mathfrak{F} be a subset of $L^{p(\cdot)}(\overline{\Omega})$, where Ω is a bounded open set in \mathbb{R}^n and let $p \in \mathcal{P}_\infty(\overline{\Omega}) \cap w\text{-Lip}(\overline{\Omega})$. The set \mathfrak{F} is relatively compact if and only if the following condition is satisfied:*

$$\lim_{h \rightarrow 0} \|f_h - f\|_{p(\cdot)} = 0 \quad \text{uniformly for } f \in \mathfrak{F}.$$

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REFERENCES

- [1] D. Cruz-Uribe SFO and A. Fiorenza, Approximate identities in variable L^p spaces. *Math. Nachr.*, **280**(03)(2007), 256–270.
- [2] L. Diening, P. Hästö and A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces. In *Function Spaces, Differential Operators and Nonlinear Analysis*, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004. Math. Inst. Acad. Sci. Czech Republic, Praha.
- [3] S. Goes; R. Welland. Compactness criteria for Köthe spaces. *Math. Ann.* **188**(1970) 251–269.
- [4] V. Kokilashvili. On a progress in the theory of integral operators in weighted Banach function spaces. In *Function Spaces, Differential Operators and Nonlinear Analysis*, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004. Math. Inst. Acad. Sci. Czech Republic, Praha.
- [5] A. Kolmogoroff, Über die Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel. *Nachr. Ges. Wiss. Göttingen*, (1931), 60–63.
- [6] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, **41**(116)(1991), 592–618.
- [7] J. Musielak. *Orlicz spaces and modular spaces*. Lecture Notes in Mathematics, 1034. Springer-Verlag, Berlin, 1983.

- [8] I.P. Natanson. *Theory of functions of a real variable. Vol. II.* Frederick Ungar Publishing Co., New York 1961.
- [9] M. Růžička. *Electrorheological Fluids: Modeling and Mathematical Theory.* Springer, Lecture Notes in Math., vol. 1748, 2000.
- [10] S. Samko. Differentiation and integration of variable order and the spaces $L^{p(x)}$. Proceed. of Intern. Conference “Operator Theory and Complex and Hypercomplex Analysis”, 12–17 December 1994, Mexico City, Mexico, Contemp. Math., Vol. 212, 203–219, 1998.
- [11] S.G. Samko, On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integr. Transf. and Spec. Funct.*, **16(5-6)**(2005), 461–482.
- [12] V. N. Sudakov, Criteria of compactness in function spaces. (Russian) *Uspehi Mat. Nauk (N.S.)* **12**(1957), no. 3(75), 221–224.
- [13] T. Takahashi, On the compactness of the function-set by the convergence in the mean of general type. *Studia Math.* **5**(1934), 141–150.
- [14] J.D. Tamarkin, On the compactness of the space L_p . *Bull. Amer. Math. Soc.* **38**(1932), 79–84.
- [15] A. Tulajkov. Zur Kompaktheit im Raum L_p für $p = 1$. *Nachr. Gess. Wiss. Göttingen, Math. Phys. Kl.* (1933), 167–170.
- [16] K. Yosida *Functional analysis*, Sixth edition. Grundlehren der Mathematischen Wissenschaften, 123. Springer-Verlag, Berlin-New York, 1980.

UNIVERSIDADE DO ALGARVE, DEPARTAMENTO DE MATEMÁTICA, CAMPUS DE GAMBELAS 8005-139 FARO – PORTUGAL
E-mail address: hrafeiro@ualg.pt